direction cosines will be the proper thermodynamic variables.[22] Consider a simple cubic ferromagnetic material in which the exchange integral is isotropic.

$$\mathcal{H}_s = -2J\sum_{ij} \vec{S}_i \cdot \vec{S}_j$$

If the spin directions change gradually so that adjacent spins differ by small angles, the quantum spin operators may be replaced by classical angular momentum vectors.

$$E = -2JS^2 \sum_{ij} \vec{\alpha}_i \cdot \vec{\alpha}_j$$

where α_j is a unit vector in the direction of spin j.

$$\vec{\alpha}_i \cdot \vec{\alpha}_j = \cos\phi_{ij}$$

may be expanded giving

$$E = -2JS^2 \sum_{ij} (1 - \tfrac{1}{2}(\vec{\alpha}_i - \vec{\alpha}_j)^2)$$

$$\simeq -2JS^2 \sum_{ij} (1 - \tfrac{1}{2}(\vec{r}_{ij} \cdot \vec{\nabla\alpha})^2),$$

where $\vec{r}_{ij}$ is a vector between magnetic ions having spin $\vec{S}_i$ and $\vec{S}_j$ and $\vec{\alpha}_i$ is extended to a continuous function of position. If only nearest neighbor interactions are assumed and the sum is extended over six nearest neighbors,

$$E \simeq -2JS^2 \sum_i \left\{6 - a^2\left[\left(\frac{\partial\vec{\alpha}}{\partial x}\right)^2 + \left(\frac{\partial\vec{\alpha}}{\partial y}\right)^2 + \left(\frac{\partial\vec{\alpha}}{\partial z}\right)^2\right]\right\}.$$

Other terms in the sum are zero because of cubic symmetry. Dropping the constant term and allowing N magnetic atoms per unit volume, the energy density becomes

$$\mathcal{E}_{ex} \simeq 2JS^2a^2N\left[\left(\frac{\partial\vec{\alpha}}{\partial x}\right)^2 + \left(\frac{\partial\vec{\alpha}}{\partial y}\right)^2 + \left(\frac{\partial\vec{\alpha}}{\partial z}\right)^2\right].$$

This suggests that in more general cases the functional dependence of the exchange energy will be

$$\mathcal{E}_{ex}\left(\frac{\partial\vec{\alpha}}{\partial x}, \frac{\partial\vec{\alpha}}{\partial y}, \frac{\partial\vec{\alpha}}{\partial z}\right).$$

In ferrimagnetic materials, which include spinel structures such as nickel ferrite and manganese zinc ferrite and garnet structures such as yttrium iron garnet, this exchange phenomenon becomes somewhat more complicated. The complication arises from the existence of diamagnetic cations regularly dispersed throughout the lattice. The exchange interaction between magnetic anions is coupled through these diamagnetic cations. Due to the large separation of the magnetic ions, there is smaller overlap of state functions and the exchange integral is negative. This type of exchange exhibited in ferrimagnetic materials is called superexchange and results in an antiferromagnetic alignment of electron spins.[24] It is found, however, that, as in the ferromagnetic case, the total Hamiltonian can again be conveniently replaced by an effective spin Hamiltonian. But, each different magnetic sublattice must be treated separately. We may still expect that a macroscopic expression for the exchange energy will be functionally dependent on the magnetization gradients.

It is observed in ferromagnetic materials that under zero applied field magnetic domains lie along preferred crystal directions.[25] Work must